THE CHINESE UNIVERSITY OF HONG KONG
MATH3270B
HOMEWORK1 SOLUTION

Question 1:

(1) We multiply t on the both sides of the equation, then the new equation can be written as:

$$\frac{d}{dt}(t^5 y) = te^{-t},$$

integral on the both sides, we get

$$t^5 y = -te^{-t} - e^{-t} + c,$$

where the constant $c = 0$ by initial data. Hence, $y = \dfrac{-te^{-t} - e^{-t}}{t^5}$.

(2) The equation can be written as:

$$\frac{d}{dt}(\sin t \cdot y) = 2e^t,$$

integral on the both sides, we get the solution:

$$y = \frac{2e^t + a\sin(1) - 2e}{\sin t}.$$

Question 2: This is a separable equation, we separate the equation, integral on the both sides and use the initial data to get the solution:

$$y^3 - 6y^2 = x + x^3 - 5.$$

Now since $y^3 - 6y^2 \in (-\infty, +\infty)$, i.e. can take all values on the real line. Hence it's sufficient to ensure $3y^2 - 12y \neq 0$, that is $y \neq 0, 4$. We substitute to the solution to find the valid interval of solution is $(-\infty, +\infty)\backslash\{x_1, x_2\}$, where:

$$x_1 \; is\ the\ solution\ of\; x^3 + x - 5 = 0,$$

$$x_2 \; is\ the\ solution\ of\; x^3 + x - 5 = -32.$$

Notice that these two values are the points of the tangent lines are vertical.(See the figure bel- low)

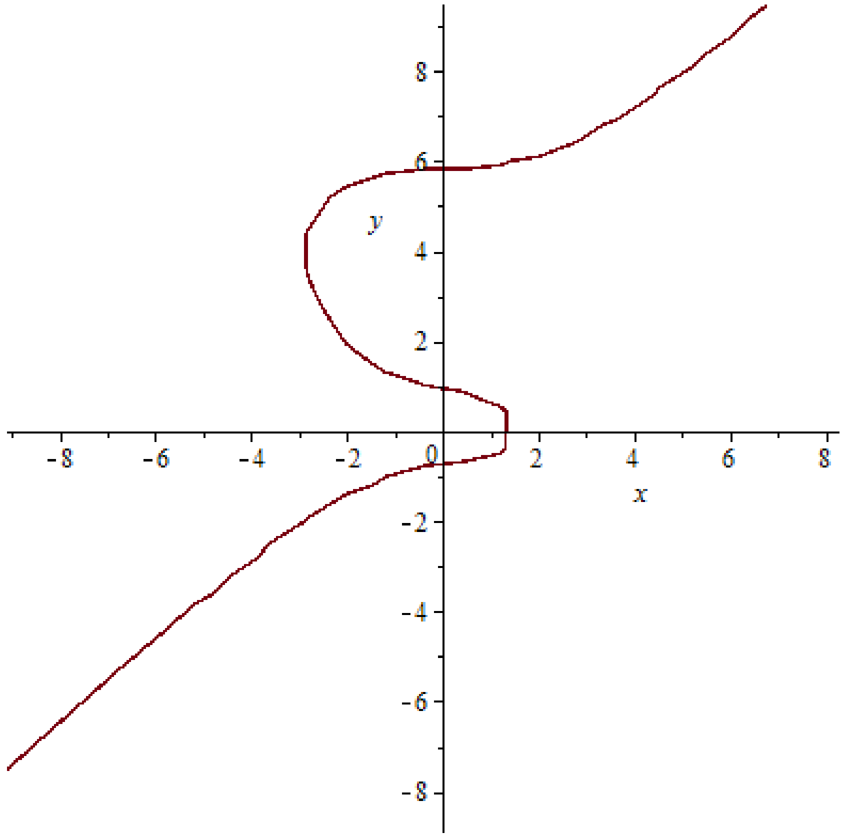

Figure 1: $y^3 - 6y^2 = x + x^3 - 5$.

Question 3:

(1) We solve this equation with following cases:

(a) If $a = b = 0$, the solution is $y = K$, where K is a constant and $K \neq -\frac{d}{c}$.

(b) In this case, we consider x is a function of y, then we get:

$$\frac{dx}{dy} = \frac{cy + d}{ay + b}, \tag{1}$$

we divide this into two cases.

(i) If $a = 0, b \neq 0$, the (1) becomes $\frac{dx}{dy} = \frac{cy + d}{b}$. Hence, the solution is $x = \frac{c}{2b}y^2 + \frac{d}{b}y + K$, where K is a constant.

(ii) If $a \neq 0$, then the (1) becomes:

$$\frac{dx}{dy} = \frac{c}{a} + \frac{ad - bc}{a}\frac{1}{ay + b},$$

integral on the both sides, we get:

$$x = \frac{c}{a}y + \frac{ad - bc}{a^2}\ln|ay + b| + K \quad (ay + b \neq 0),$$

where K is a constant.

(2) This question is wrong since the initial data cannot ensure the existence of solution, and we change the initial data to be $y(1) = 1$. We multiply y on the both sides of the equation, then the equation becomes

$$y^2dt + (2ty - 3y^2e^y)dy = 0 \tag{2}$$

one can check that (2) is an exact equation. Hence the solution is $ty^2 - (3y^2e^y - 6ye^y + 6e^y) = 1 - 3e$.

Question 4:

(1) Here $M_y = e^x \cos y - 3\sin x$, $N_y = e^x \cos y - 3\sin x$, hence this equation is exact, and the solution is

$$e^x \sin y + 3y\cos x = c,$$

where c is a constant.

(2) In this equation, $M_y = \frac{x}{y} + x$, $N_y = \frac{y}{x} + y$. Hence this equation is not exact.

(3) Here $M_y = \frac{3xy(x^2+y^2)^{1/2}}{(x^2+y^2)^3}$, $N_x = \frac{3xy(x^2+y^2)^{1/2}}{(x^2+y^2)^3}$. Hence, this equation is exact and the solution is

$$(x^2 + y^2)^{-1/2} = c,$$

where c is a constant.

Question 5:

If $Mdx + Ndy = 0$ has an integrating factor μ, then we have $(\mu M)_y = (\mu N)_x$, i.e.

$$\mu(N_x - M_y) = M\mu_y - N\mu_x \tag{3}$$

Now if μ has a form of $\mu(xy)$, then we deduce that $\mu_x = y\mu'$, $\mu_y = x\mu'$. Then the (3) becomes $\mu' = R\mu$. Finally, we deduce that $\mu = Ce^{\int R}$, where C is a constant.